Skolem Meets Schanuel

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Skolem and Positivity Problems: Classical Formulation

A **linear recurrence sequence (LRS)** is a sequence in $\mathbb{Z}$ (or $\mathbb{Q}$) $\langle u_0, u_1, u_2, \ldots \rangle$ such that there are constants $a_1, \ldots, a_k$ and,

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Problem POSITIVITY (mid-1970s)

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Question: Is it the case that, $\forall n \geq 0$, $u_n \geq 0$?

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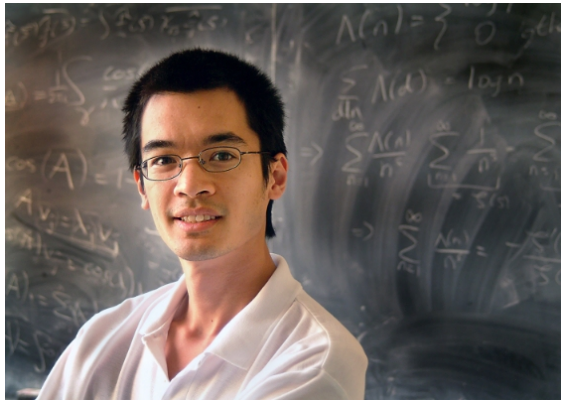

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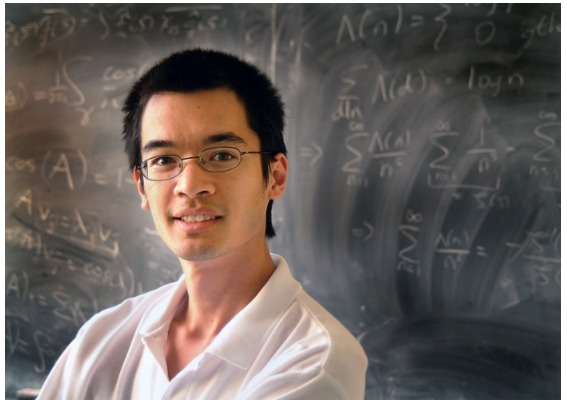

"A mathematical embarrassment . . ."

"Arguably, by some distance, the most prominent problem whose decidability status is currently unknown."

Richard Lipton

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- Decidability of the Skolem Problem is equivalent to being able to compute the finite set of zeros of any given non-degenerate LRS
- Unfortunately, all known proofs of the Skolem-Mahler-Lech Theorem make use of *non-constructive* p-adic techniques

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$x \equiv \langle 1, 2, 1, 2, 1, 2, \ldots \rangle \pmod 3$
$y \equiv \langle 0, 0, 0, 0, 0, 0, \ldots \rangle \pmod 3$
$z \equiv \langle 0, 0, 0, 0, 0, 0, \ldots \rangle \pmod 3$

The Fibonacci Recurrence: $u_{n+2} = u_{n+1} + u_n$

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$\langle 2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, \ldots \rangle$

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$\Rightarrow$ **Never zero!**

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- The shifted Fibonacci sequence doesn't contain a zero, but is haunted by the ghost of a zero *in its past!*

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Simple LRS correspond precisely to **diagonalisable** matrices

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Theorem (O. & Worrell 2014)

- *For LRS of order ≤ 5, POSITIVITY is decidable.*
- *For simple LRS of order ≤ 9, POSITIVITY is decidable.*
- *For LRS of order ≥ 6, POSITIVITY is hard with respect to longstanding Diophantine-approximation problems.*

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- Many, many results subject to P$\neq$NP, or ETH, etc...

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$$u_{n+k} = a_1 u_{n+k-1} + \ldots + a_k u_n$$

Then

$$u_n = \frac{-1}{a_k}\left(a_{k-1} u_{n+1} + a_{k-2} u_{n+2} + \ldots + a_1 u_{n+k-1} - u_{n+k}\right)$$

So if a_k is invertible (mod m), the entire bi-infinite sequence is well-defined in $\mathbb{Z}/m\mathbb{Z}$. Therefore we require that $\gcd(m, a_k) = 1$.

- Example: $u_{n+1} = 2u_n$:

$$\left\langle \ldots, \frac{1}{32}, \frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8, 16, 32, \ldots \right\rangle$$

$$\langle \ldots, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, \ldots \rangle \pmod 3$$

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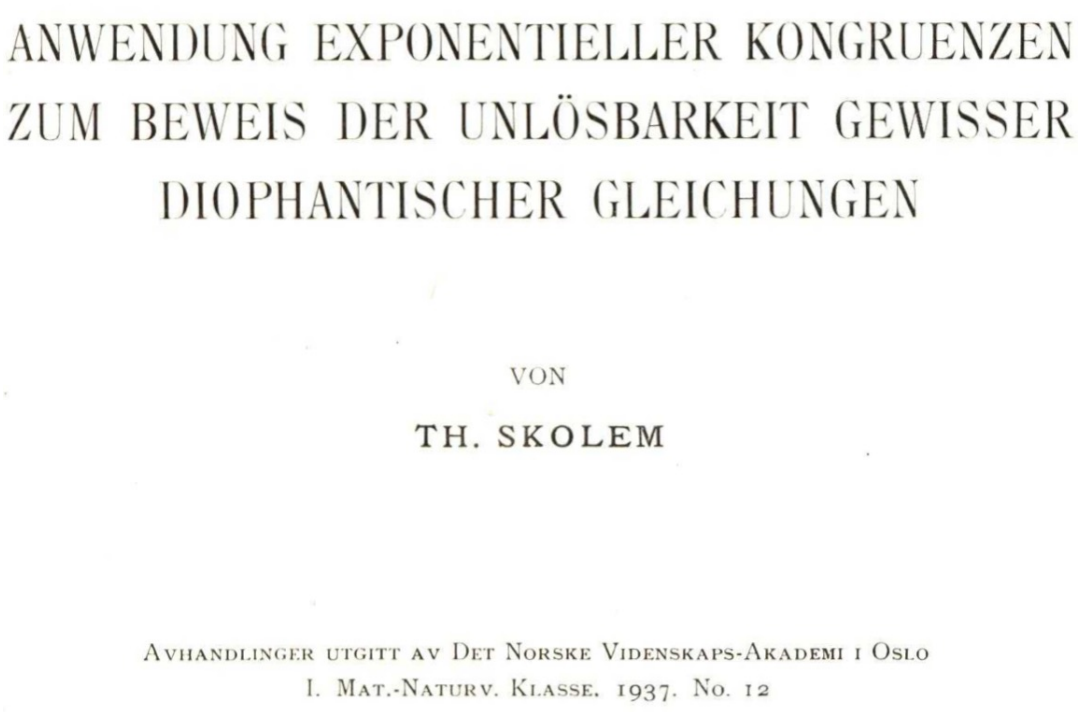
The Skolem Conjecture



ANWENDUNG EXPONENTIELLER KONGRUENZEN
ZUM BEWEIS DER UNLÖSBARKEIT GEWISSER
DIOPHANTISCHER GLEICHUNGEN

VON

TH. SKOLEM

AVHANDLINGER UTGITT AV DET NORSKE VIDENSKAPS-AKADEMI I OSLO
I. MAT.-NATURV. KLASSE. 1937. NO. 12

- A fairly wide-ranging conjecture, formulated in 1937, also known as the **Exponential Local-Global Principle**
- Like Schanuel's Conjecture, widely believed by number theorists, but only proven in special cases

The Skolem Conjecture

The Skolem Conjecture for simple bi-LRS (1937)

Consider the recurrence equation $u_{n+k} = a_1 u_{n+k-1} + \ldots + a_k u_n$, with $u_0, \ldots, u_{k-1}, a_1, \ldots, a_k \in \mathbb{Z}$. Suppose the bi-LRS $\langle u_n \rangle_{n=-\infty}^{\infty}$ is simple. Then $\langle u_n \rangle_{n=-\infty}^{\infty}$ has no zeros iff, for some integer $m \geq 2$ with $\gcd(m, a_k) = 1$, we have that for all $n \in \mathbb{Z}$, $u_n \not\equiv 0 \pmod{m}$.

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Equivalently:

If a simple bi-infinite LRS over the rationals has no zeros, then this will necessarily be witnessed modulo *some* integer m.

The Skolem Problem for Simple LRS

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Theorem (Bilu, Luca, Nieuwveld, O., Purser, Worrell 2022)

There is an algorithm which takes as input a simple, non-degenerate LRS and produces its (finite) set of zeros.

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Accounts Teams

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$a_1\ a_2\ \ldots\ a_k$
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where:

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Input area

Auto-fill examples: Show/Hide

Zero LRS | Degenerate LRS | Non-simple LRS | Trivial | Fibonacci | Tribonacci | Berstel sequence [1] | Order 5 [3] | Order 6 [3] | Reversible order 8 [3]

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```
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0 0 -48 -120 0 520 624 -2016
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- [x] Always render full LRS (otherwise restricted to 400 characters)
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Go | Clear | Stop

Output area

Zeros: 0, 1, 4

Zero at 0 in $(0+ 1\mathbb{Z})$ hide/show

- p-adic non-zero in $(0+ 136\mathbb{Z}_{\neq 0})$
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  - p-adic non-zero in $(1+ 680\mathbb{Z}_{\neq 0})$ ($(0+ 5\mathbb{Z}_{\neq 0})$ of parent)
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  - Non-zero mod 3 in $(273+ 680\mathbb{Z})$ ($(2+ 5\mathbb{Z})$ of parent)
  - Non-zero mod 9 in $(409+ 680\mathbb{Z})$ ($(3+ 5\mathbb{Z})$ of parent)
  - Non-zero mod 3 in $(545+ 680\mathbb{Z})$ ($(4+ 5\mathbb{Z})$ of parent)
- Non-zero mod 7 in $(2+ 136\mathbb{Z})$

```
================
LRS: u_{n} =
-2716131161712097448586635205589463470401509550890641913636335454675409769151
1} +
-5087571794255306084649276133206965823971875016365294395124753570723932449955
2} +
-1020664001586411899151994265194472024922159984096674355479305686778200805260
3} +
-1412095662406000310364496715181260667298901575064822931268517590804654375986
4} +
1906955894773207103609842658940914223756942339091587019654461069437273467027
5} +
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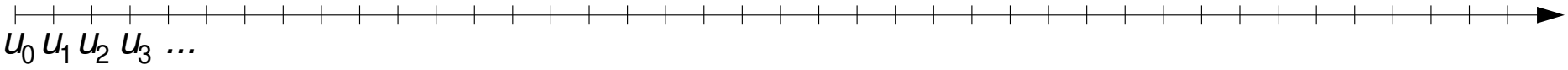

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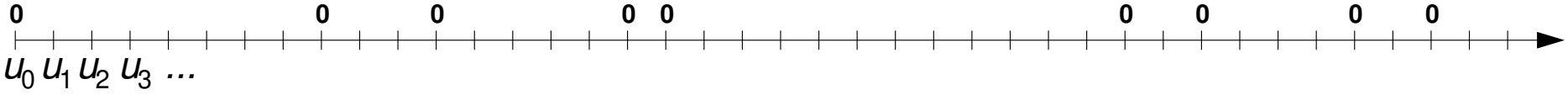

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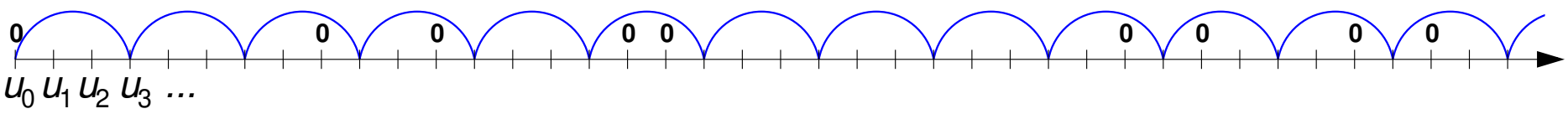

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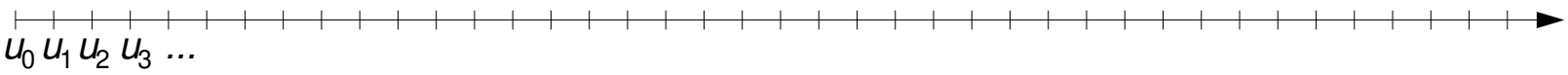

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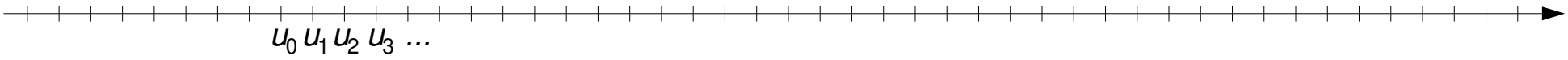

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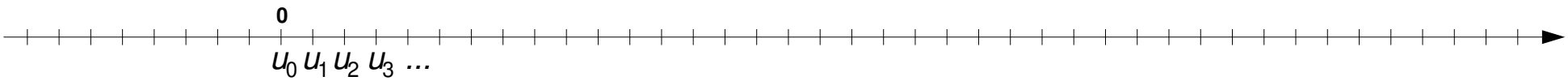

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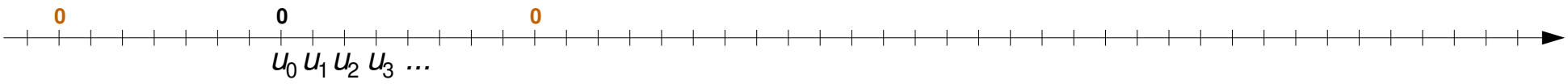

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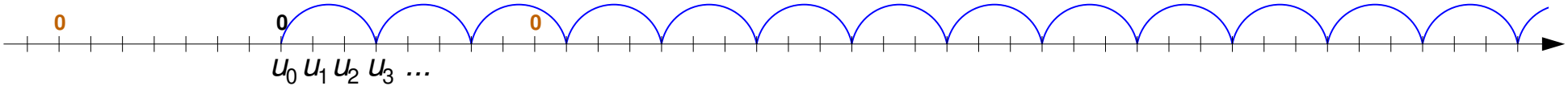

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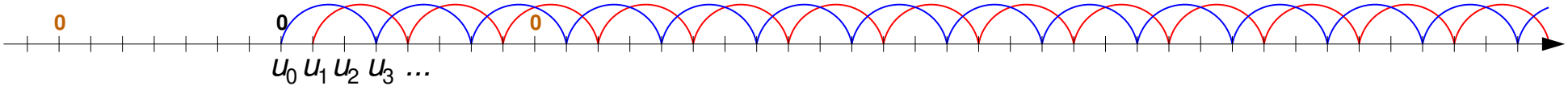

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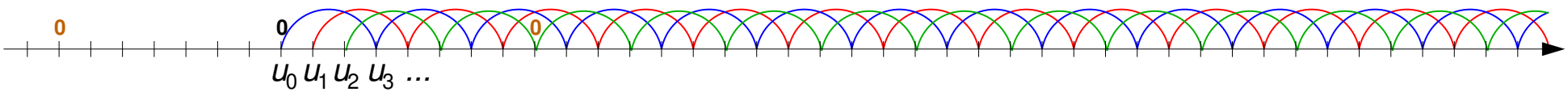

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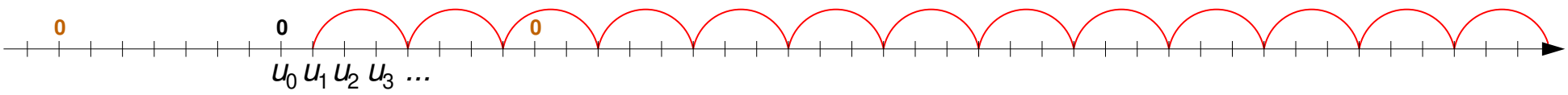

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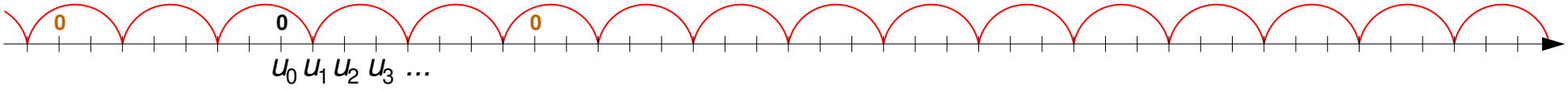

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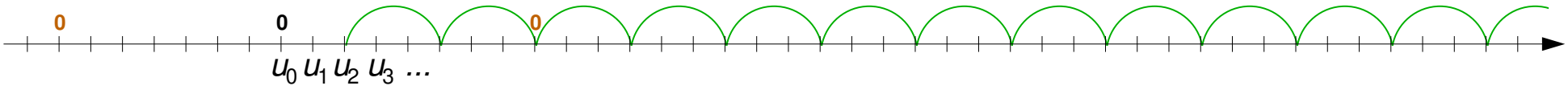

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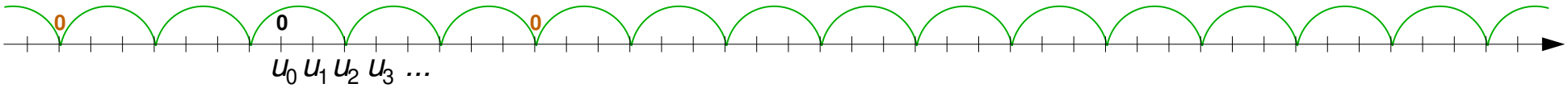

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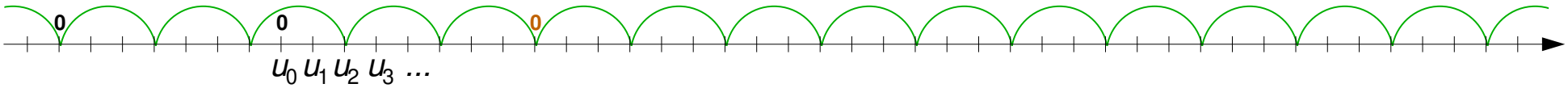

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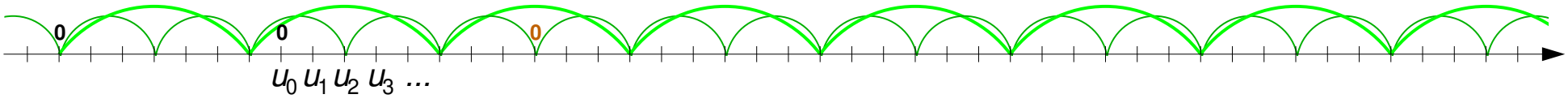

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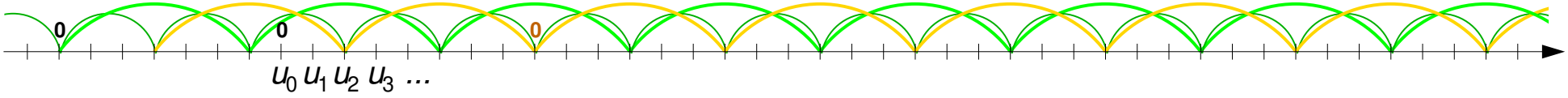

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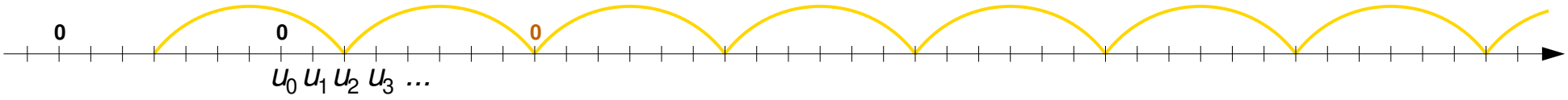

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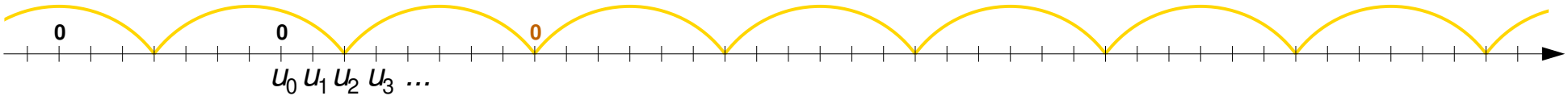

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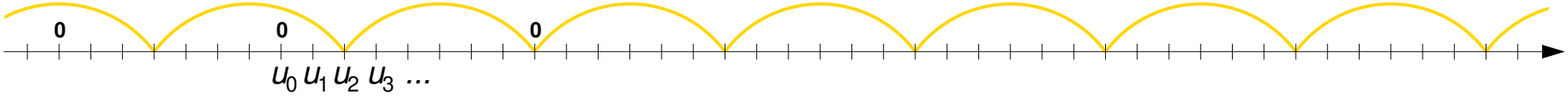

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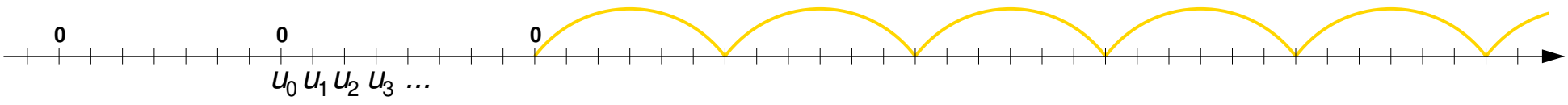

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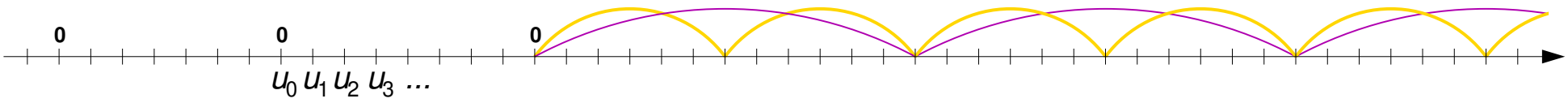

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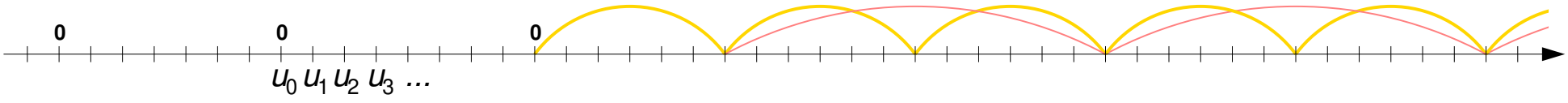

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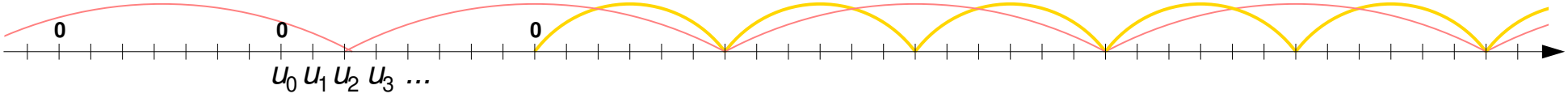

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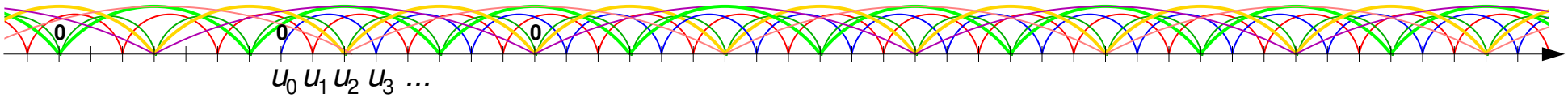

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- p-adic non-zero in $(0+ 136\mathbb{Z}_{\neq 0})$
- Zero at 1 in $(1+ 136\mathbb{Z})$ hide/show
  - p-adic non-zero in $(1+ 680\mathbb{Z}_{\neq 0})$ ($(0+ 5\mathbb{Z}_{\neq 0})$ of parent)
  - Non-zero mod 3 in $(137+ 680\mathbb{Z})$ ($(1+ 5\mathbb{Z})$ of parent)
  - Non-zero mod 3 in $(273+ 680\mathbb{Z})$ ($(2+ 5\mathbb{Z})$ of parent)
  - Non-zero mod 9 in $(409+ 680\mathbb{Z})$ ($(3+ 5\mathbb{Z})$ of parent)
  - Non-zero mod 3 in $(545+ 680\mathbb{Z})$ ($(4+ 5\mathbb{Z})$ of parent)
- Non-zero mod 7 in $(2+ 136\mathbb{Z})$

```
================
LRS: u_{n} =
-2716131161712097448586635205589463470401509550890641913636335454675409769151
1} +
-5087571794255306084649276133206965823971875016365294395124753570723932449955
2} +
-1020664001586411899151994265194472024922159984096674355479305686778200805260
3} +
-1412095662406000310364496715181260667298901575064822931268517590804654375986
4} +
1906955894773207103609842658940914223756942339091587019654461069437273467027
5} +
```

The Skolem Landscape

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