# The Post Correspondence Problem: from Computer Science to Algebra

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The Post Correspondence Problem in computer science

# PCP in the computer science books



## 5.2

## A SIMPLE UNDECIDABLE PROBLEM

In this section we show that the phenomenon of undecidability is not confined to problems concerning automata. We give an example of an undecidable problem concerning simple manipulations of strings. It is called the ***Post correspondence problem***, or ***PCP***.

We can describe this problem easily as a type of puzzle. We begin with a collection of dominos, each containing two strings, one on each side. An individual domino looks like

$$\left[\frac{\texttt{a}}{\texttt{ab}}\right]$$

and a collection of dominos looks like

$$\left\{ \left[\frac{\texttt{b}}{\texttt{ca}}\right], \left[\frac{\texttt{a}}{\texttt{ab}}\right], \left[\frac{\texttt{ca}}{\texttt{a}}\right], \left[\frac{\texttt{abc}}{\texttt{c}}\right] \right\}$$

The task is to make a list of these dominos (repetitions permitted) so that the string we get by reading off the symbols on the top is the same as the string of symbols on the bottom. This list is called a ***match***. For example, the following list is a match for this puzzle.

$$\left[\frac{\texttt{a}}{\texttt{ab}}\right]\left[\frac{\texttt{b}}{\texttt{ca}}\right]\left[\frac{\texttt{ca}}{\texttt{a}}\right]\left[\frac{\texttt{a}}{\texttt{ab}}\right]\left[\frac{\texttt{abc}}{\texttt{c}}\right].$$

## PCP in algebra: equalisers of maps

Let $P, Q$ be two groups or monoids.

### Definition

The equaliser or equality language of two maps $g, h : P \to Q$ is

$$\boxed{\mathrm{Eq}(g, h) := \{x \in P \mid g(x) = h(x)\}.}$$

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## Statement

*The Post Correspondence Problem* PCP *is the decision problem asking:*

$$\boxed{\text{Is } \mathsf{Eq}(g,h) \text{ trivial?}}$$

# The classical Post Correspondence Problem

Let $P = \Sigma^*, Q = \Delta^*$ be two free monoids.

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The Post Correspondence Problem PCP is the decision problem:

Given morphisms $g, h : \Sigma^* \to \Delta^*$, is the equaliser $\mathsf{Eq}(g, h)$ trivial ?

Solutions to the instance $I = (\Sigma, \Delta, g, h)$ are words in $\mathsf{Eq}(g, h) \setminus \{\epsilon\}$.

# The classical Post Correspondence Problem

For example, take $\Sigma = \{y, z\}$, $\Delta = \{a, b\}$, and $g, h : \Sigma^* \mapsto \Delta^*$

$$1. \quad \begin{aligned} g : y &\mapsto a \\ z &\mapsto a \end{aligned} \qquad \begin{aligned} h : y &\mapsto a^2 \\ z &\mapsto a^2 \end{aligned}$$

---

$$2. \quad \begin{aligned} g : y &\mapsto ab \\ z &\mapsto b \end{aligned} \qquad \begin{aligned} h : y &\mapsto a \\ z &\mapsto b^2 \end{aligned}$$

# The classical Post Correspondence Problem

**Theorem (Post, 1946)**

*The Post Correspondence Problem in free monoids is* **undecidable**.

Several proofs:

1. reduce PCP to the Halting Problem in Turing Machines.
2. reduce PCP to the Word Problem in semigroups.

# Applications of PCP

**Theorem (Halava, Harju, 2001)**

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### Theorem

*(1) It is undecidable whether a context-free grammar is ambiguous.*

*(2) It is undecidable for arbitrary context-free grammars $G_1$ and $G_2$ whether*

- $L(G_1) \cap L(G_2) = \emptyset$.
- $L(G_1) \subseteq L(G_2)$
- $L(G_1)$ *is equal to some regular language* $R$
- ...

# Always undecidable?

Let PCP($n$) denote all instances $I = (\Sigma, \Delta, g, h)$ with $|\Sigma| = n$.

| | |
|---|---|
| PCP(2) | |
| PCP(3) | |
| PCP(4) | |
| PCP(5) | |

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**Theorem (Lecerf, 1963)**

*The* PCP *is still undecidable when $g$ and $h$ are injective.*

## The Post Correspondence Problem in free groups

# Free groups

$F(X) =$ free group on generating set $X$.

Example

$F(a, b) =$

- the set of all *reduced* words on $a, a^{-1}, b, b^{-1}$
- with product $\circ$ given by concatenation and free reduction:

$$aba^{-1} \circ ab^2a = ab^3a,$$

- identity element 1 (although it corresponds to the empty word $\epsilon$).

# The PCP for free groups

- Statement

  *The PCP for free group morphisms is the decision problem:*

  $$\textit{Given } g, h : F(\Sigma) \to F(\Delta), \textit{ is the equaliser } \mathsf{Eq}(g, h) \textit{ trivial?}$$

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$$\text{Given } g, h : F(\Sigma) \to F(\Delta), \text{ is the equaliser } \mathsf{Eq}(g, h) \text{ trivial?}$$

- Question

*Is the PCP in free groups decidable? WE DON'T KNOW!*

# The PCP for free groups and free monoids: literature

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- $\sim$ 20 papers and preprints for free groups – about algorithms
- $>$ 50 papers for free groups – about geometry & topology connections

# Equalisers of free group homomorphisms

## Definition

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**Exercise**: $\mathrm{Eq}(g, h)$ is a subgroup of $F(\Sigma)$.

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*PCP: Is the subgroup* $\mathrm{Eq}(g, h)$ *trivial?*

*PCP+: Is the subgroup* $\mathrm{Eq}(g, h)$ *finitely generated?*

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*In general,* $\mathrm{Eq}(g, h)$ *is NOT finitely generated.*

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## Theorem (Goldstein+Turner, 1986)

* $\mathrm{Eq}(g, h)$ *is finitely generated if at least one of g or h is injective.*

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## Theorem (Goldstein+Turner, 1986)

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*MATHSCINET: The proof uses Nielsen reduced generators and cancellation arguments expressed in a delightfully simple graphical form.

# Equalisers of free group morphisms

Consider $g, h : F(\Sigma) \to F(\Delta)$ with $g$ injective.

Question (Equaliser conjecture – Stallings, 1984)

*Does the subgroup* $\mathrm{Eq}(g, h)$ *always have* $\leq |\Sigma|$ *generators?*

# Equalisers of free group morphisms

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Question (Equaliser conjecture – Stallings, 1984)

*Does the subgroup* $\mathrm{Eq}(g, h)$ *always have* $\leq |\Sigma|$ *generators?*

Question (Basis Problem – Stallings, 1984)

*PCP++: Is there an algorithm to compute a basis (i.e. set of generators) for the subgroup* $\mathrm{Eq}(g, h)$*?*

# Fixed subgroups: a special case

For $id, \phi : F(\Sigma) \to F(\Sigma)$, the equaliser consists of the fixed points of $\phi$:

$$\boxed{\mathrm{Eq}(id, \phi) = \mathrm{Fix}(\phi) = \{x \in F(\Sigma) \mid \phi(x) = x\}.}$$

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Vast literature on $\mathrm{Fix}(\phi)$ if $\phi$ is an automorphism of a free group: Gersten, Bestvina, Feighn, Handel, Bogopolski, Maslakova, ...

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Theorem (C.& Logan, 2020)

*There is an algorithm to compute a basis for* $\mathrm{Fix}(\phi)$, *if* $\phi$ *is an endomorphism of* $F_2$, *the free group on two generators.*

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### Theorem (JP Mutanguha, 2021)

*There is an algorithm to compute a basis for* $\mathrm{Fix}(\phi)$*, if* $\phi$ *is an endomorphism of* $F_k$*, the free group on* $k$ *generators,* $k \geq 2$*.*

## Back to the classical PCP: free monoid homomorphisms

The equaliser or equality language of morphisms $g, h : \Sigma^* \to \Delta^*$ is

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*Output a finite automaton recognising* $\mathrm{Eq}(g,h)$*?*

## Answer

*Finding an automaton implies PCP is decidable, so* **no algorithm exists***:*

1. *A finite automaton may not exist.*
2. *Even when an automaton exists, there is no algorithm to produce it (Karhumäki+Saarela, 2010).*

## The PCP for free groups: questions

1. Can we use free monoid ideas/techniques to resolve PCP in free groups?
2. Does PCP relate to Stallings' Basis Problem:

   Given $g, h : F(\Sigma) \to F(\Delta)$ with $g$ injective, find a basis for $\mathrm{Eq}(g, h)$.

3. Are there

   (i) types of maps or

   (ii) variations of the PCP

   where we can make progress?

## Variations and subcases of the PCP

# Types of maps: injectivity

PCP: Given morphisms $g, h : F(\Sigma) \to F(\Delta)$ is $\mathrm{Eq}(g, h)$ trivial?

- $g, h$ both injective

  ???

- one of $g, h$ injective

  ???

- neither $g, h$ injective

  PCP is **decidable** if neither map is injective:

$$\{x^{-1}y^{-1}xy \mid x \in \ker(g), y \in \ker(h)\} \subseteq \mathrm{Eq}(g, h).$$

## The PCP for free groups and free monoids: status of results

| **Problems** | **In free monoids** | **In free groups** |
|---|---|---|
| **general PCP** | undecidable | unknown |
| **Basis/Equaliser Problem** | undecidable | unknown |
| **PCP non-injective** | undecidable | decidable |
| **PCP injective** | undecidable | unknown |

# The PCP for free groups and free monoids: positive results

| Problems | In free monoids | In free groups |
|---|---|---|
| marked PCP | decidable | decidable |
| | Halava+Hirvensalo+de Wolf, 2001 | C. - Logan, 2020 |
| generic PCP | decidable | decidable |
| | Gilman+Miasnikov+Miasnikov+Ushakov'06 | C.+Martino+Ventura'08 |

# Marked morphisms and immersions I

A set of words $\mathbf{S} \subseteq \Delta^+$ is marked if each $u \in \mathbf{S}$ starts with a different letter.

## Example

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## Free monoids

*A free monoid morphism $f : \Sigma^* \to \Delta^*$ is marked if the set $f(\Sigma) \subset \Delta$ is marked.*

## Example

$$f : \{a, b\}^* \to \{x, y\}^*$$

$$a \mapsto xy$$

$$b \mapsto yxy$$

This is marked as $f(\{a, b\}) = \{xy, yxy\}$.

# Marked morphisms and immersions II

## Marked morphisms

- Central to the proof of the PCP(2)

  (Ehrenfeucht+Karhumäki+Rozenberg, 1982)
- Marked PCP is decidable

  (Halava+Hirvensalo+de Wolf, 2001)
- The density of marked morphisms among all morphisms $\Sigma^* \to \Delta^*$ is a positive constant, dependent on $|\Delta|$ and $|\Sigma|$.

  (C.+Logan, 2020)

# Marked morphisms and immersions III

An immersion of free groups is a morphism $f : F(\Sigma) \to F(\Delta)$ where the set $\mathbf{f}(\boldsymbol{\Sigma} \sqcup \boldsymbol{\Sigma}^{-1}) \subset F(\Delta)$ is marked.

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## Example

$$f : F(a, b) \to F(x, y)$$

$$a \mapsto xy$$

$$b \mapsto yxy$$

This is not an immersion as:

$$f(\{a, b, a^{-1}, b^{-1}\}) = \{xy, yxy, y^{-1}x^{-1}, y^{-1}x^{-1}y^{-1}\}.$$

# Main theorem: marked PCP

Inspired by the proof of the marked PCP, we prove:

## Theorem (C.-Logan, 2020)

*If* $g, h : F(\Sigma) \to F(\Delta)$ *are immersions then there exists an alphabet* $\Sigma_{g,h}$ *and an immersion* $\psi : F(\Sigma_{g,h}) \to F(\Sigma)$ *such that*

$$\mathrm{im}(\psi) = \mathrm{Eq}(g, h).$$

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### Corollary

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### Corollary

*If $g, h : F(\Sigma) \to F(\Delta)$ are immersions then $\mathrm{rank}(\mathrm{Eq}(g, h)) \leq |\Sigma|$.*

# GPCP = the generalised PCP

Given alphabets $\Sigma, \Delta$, maps $g, h : F(\Sigma) \to F(\Delta)$ and words $u_1, u_2, v_1, v_2 \in F(\Delta)$, is there $x \in F(\Sigma) \setminus \{1\}$ such that

$$\boxed{u_1 g(x) u_2 = v_1 h(x) v_2?}$$

# The GPCP for free groups and free monoids: status of results

| Problems | In free monoids | In free groups |
|---|---|---|
| GPCP | undecidable<br>Harju+Karhumäki'97 | undecidable<br>Myasnikov+Nikolaev+Ushakov'14 |
| GPCP non-injective | undecidable<br>Harju+Karhumäki'97 | undecidable<br>Myasnikov+Nikolaev+Ushakov'14 |
| GPCP injective | undecidable<br>Lecerf'63 | unknown |

## Connections in free groups

$$\begin{tikzcd}[column sep=large]
 & & \mathsf{PCP} \arrow[d, Rightarrow] \\
\text{Basis Problem} \arrow[r, Rightarrow] & \text{Rank Problem} \arrow[r, Rightarrow] & \mathsf{PCP}^{\mathsf{inj}} \\
 & & \mathsf{GPCP}^{\mathsf{inj}} \arrow[u, Rightarrow]
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**Theorem (C.+Logan, 2021)**

Basis Problem $\Rightarrow$ $\mathsf{GPCP}^{\mathsf{inj}}$

## Connections in free groups

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**Theorem (C.+Logan, 2021)**

Basis Problem $\Rightarrow$ GPCP$^{\text{inj}}$

**Theorem (C.+Logan, 2021)**

Basis Problem $\Leftrightarrow$ Rank Problem.

$$\text{Basis Problem} \Longleftrightarrow \text{Rank Problem} \Longrightarrow \text{GPCP}^{\text{inj}} \Longrightarrow \text{PCP}$$

## 1st recap: free monoids and groups

- PCP & GPCP are both undecidable for free monoids.
- So are the injective versions for free monoids: $\mathrm{PCP}^{\mathrm{inj}}$ & $\mathrm{GPCP}^{\mathrm{inj}}$.

## 1st recap: free monoids and groups

- PCP & GPCP are both undecidable for free monoids.
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- PCP , $\mathrm{GPCP}^{\mathrm{inj}}$ and Stallings' Rank Problem remain open for free groups.

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- So are the injective versions for free monoids: $\mathrm{PCP}^{\mathrm{inj}}$ & $\mathrm{GPCP}^{\mathrm{inj}}$.
- The GPCP is undecidable for free groups.
- PCP, $\mathrm{GPCP}^{\mathrm{inj}}$ and Stallings' Rank Problem remain open for free groups.
- The Basis Problem and Stallings' Rank Problem are equivalent.
- We have a nice sequence of implications:

$$\text{Rank Problem} \Longrightarrow \mathrm{GPCP}^{\mathrm{inj}} \Longrightarrow \mathrm{PCP}$$

# PCP in other groups

# Redefining PCP

Is PCP (i.e. the triviality of equalisers) the right problem in an abelian group $\Gamma$?

Example

Let $g, h : F(\Sigma) \to \Gamma$ be two homomorphisms. Then for any $a, b \in F(\Sigma)$

$$g(aba^{-1}b^{-1}) = h(aba^{-1}b^{-1}) = \mathbf{0},$$

so all commutators $aba^{-1}b^{-1}$ are solutions to PCP.

# Redefining PCP

Is PCP the right problem in the $3 \times 3$ Heisenberg group $H$?

$H$ is nilpotent of class 2 $\implies$ $[[x, y], z] = 1$ for any $x, y, z \in H$.

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THE POINT: the 'laws' of the group are solutions to PCP!

## PCP for general groups: redefining the problem

Let $\Gamma$ be a group with 'laws', such as abelian or nilpotent or Burnside etc.

<u>'Verbal' PCP:</u>

- $g, h : F(\Sigma) \to \Gamma$
- Does there exist $x \in F(\Sigma)$ satisfying $g(x) = h(x)$ non-trivially?

Is $\mathrm{Eq}(g, h) \setminus$ set of laws in $\Gamma$ non-trivial?

# PCP for general groups: results

Theorem (C. - Logan - Levine, 2022)

*The verbal PCP is decidable for (torsion-free) nilpotent groups.*

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- If neither map is injective: PCP is not very interesting!

## PCP for general groups: redefining the problem

- Let $\Gamma$ be a group without laws.
- Verbal PCP becomes simply about the triviality of equalisers.
- Statement of PCP for $\Gamma$:
  - Two homomorphisms

$$g, h : F(\Sigma) \to \Gamma$$

  - Does there exist a non-trivial $x \in F(\Sigma)$ such that $g(x) = h(x)$?

- If neither map is injective: PCP is not very interesting!

RECALL:

PCP is **decidable** if neither $g$ nor $h$ is injective:

$$\{x^{-1}y^{-1}xy \mid x \in \ker(g), y \in \ker(h)\} \subseteq \mathrm{Eq}(g, h).$$

## New statement of PCP

'Kernel' PCP:

- $g, h : F(\Sigma) \to \Gamma$
- Does there exist $x \in F(\Sigma)$ satisfying $g(x) = h(x)$ non-trivially?

$$\boxed{\text{Is } \mathrm{Eq}(g,h) \backslash (\ker(g) \cap \ker(h)) \text{ non-trivial?}}$$

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REMARK: If at least one of $g$ and $h$ is injective we get the classical PCP because

$$\ker(g) \cap \ker(h) = \{1\}.$$

# Important classes of groups and the kernel PCP

W/o laws

- Free groups
- Hyperbolic groups (eg $SL(2, \mathbb{Z})$)

With laws

- Nilpotent groups (eg the $3 \times 3$ integral Heisenberg group)

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SO FAR: no PCP undecidability results!

# Important classes of groups and the kernel PCP

W/o laws

1. Free groups: open ???

2. Hyperbolic groups: undecidable !!!

With laws

3. Nilpotent groups:

# Important classes of groups and the kernel PCP

W/o laws

1. Free groups: open ???

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With laws

3. Nilpotent groups:

   'laws of the group' $\leq \ker(g) \cap \ker(h)$, so the verbal PCP may have solutions when the kernel PCP does not

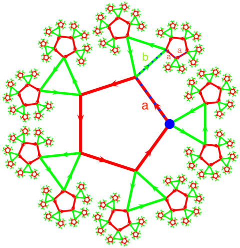
   $\Longrightarrow$

   the kernel PCP and the verbal PCP are not equivalent.

# Hyperbolic groups

**Motivation:** Most (finitely presented) groups are hyperbolic.

**Definition:** Groups whose Cayley graph looks like the hyperbolic plane.




**Examples:** free groups, free products of finite groups, $SL(2,\mathbb{Z})$, virtually free groups *, and many more.

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* Virtually free = groups with a free subgroup of finite index.

# Hyperbolic groups

Theorem (C., Logan & Levine, 2022)

*There exists a hyperbolic group with undecidable (binary) PCP.*

# Hyperbolic groups

**Theorem (C., Logan & Levine, 2022)**

*There exists a hyperbolic group with undecidable (binary) PCP.*

**Proof.**

Key ingredients:

- The subgroup membership problem in hyperbolic groups is undecidable.
- The Rips construction: can build a hyperbolic group out of given groups.
- Choose the groups above to give undecidability.

□

# Nilpotent groups and more

- Nilpotent groups: ‘kernel’ PCP is decidable in polynomial time.

  (Myasnikov, Nikolaev, Ushakov, 2014)

## Nilpotent groups and more

- Nilpotent groups: ‘kernel’ PCP is decidable in polynomial time.

  (Myasnikov, Nikolaev, Ushakov, 2014)

- Virtually nilpotent groups: ‘kernel’ PCP is decidable.

  (C., Logan, Levine, 2022)

# Final recap

Two versions of PCP in general groups:

- 'Verbal' PCP in groups with laws

  decidable in nilpotent groups

- 'Kernel' PCP in groups without laws: free, hyperbolic, virtually nilpotent

  undecidable hyperbolic examples

# Questions

- PCP in free groups?
- PCP in further groups?
- PCP in non-free monoids?
- Complexity results?

*Thank you!*