A Universal Skolem Set of Positive Lower Density

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Halting Problem

Instance: $\langle\ \mathbf{a}\ ;\ \mathbf{b}\ ;\ \mathbf{M}\ \rangle$
Question: Does this program halt?

Classical Formulation

A **linear recurrence sequence (LRS)** is a sequence $\langle u_0, u_1, u_2, \ldots \rangle$ in $\mathbb{Q}$ such that there are constants $a_1, \ldots, a_k$ and,

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Problem SKOLEM

Instance: An LRS $\langle u_0, u_1, u_2, \ldots \rangle$
Question: Does $\exists n \geq 0$ such that $u_n = 0$?

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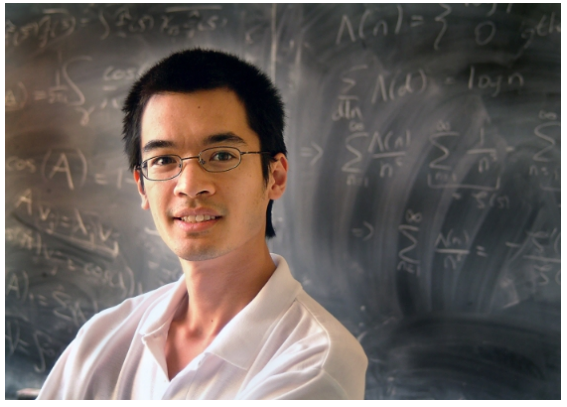
Terence Tao



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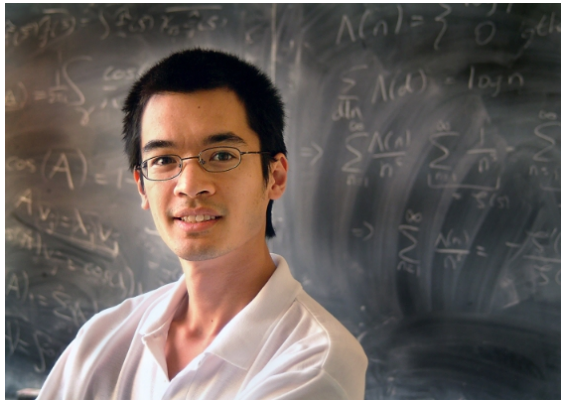

"...a mathematical embarrassment..."

Richard Lipton

The Skolem-Mahler-Lech Theorem

Theorem (Skolem 1934; Mahler 1935, 1956; Lech 1953)

Let $\langle u_n \rangle$ be a **non-degenerate** *linear recurrence sequence that is not identically zero. Then the set $\{n : u_n = 0\}$ is finite.*

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- Skolem's proof used p-adic analysis and is ineffective.
- From the early 2000s: series of papers that use the Subspace Theorem in Diophantine approximation to give explicit upper bounds on the number of zeros of a non-degenerate LRS.

The Skolem Problem at Low Orders

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Corollary

The Halting Problem is decidable for loops with at most 4 variables

Universal Skolem Set

Definition

Define $\mathcal{S} \subseteq \mathbb{N}$ to be a **Universal Skolem Set** if there is an effective procedure that inputs an integer linear recurrence sequence $\langle u_n \rangle$ and outputs whether or not there exists $n \in \mathcal{S}$ with $u_n = 0$.

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Example

Define $f : \mathbb{N}_+ \to \mathbb{N}$ by $f(n) = \lfloor \sqrt{\log n} \rfloor$. Write $s_0 := 1$ and, inductively set $s_n := n! + s_{f(n)}$ for $n > 0$. Then $\mathcal{S} := \{s_n : n \in \mathbb{N}\}$ is a Universal Skolem Set.

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- $\mathcal{S}$ has density zero:

$$|\mathcal{S} \cap \{1, \ldots, n\}| \sim \frac{\log n}{\log \log n}$$

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Theorem

The set $\mathcal{U}$ is a Universal Skolem Set of positive lower density.

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- Simple LRS correspond precisely to **diagonalisable** matrices

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Theorem (Schlickewei and Schmidt 2000)

Let α_i, β_i, C_i be non-zero algebraic numbers for $i = 1, \ldots, \ell$, Excepting some degenerate cases, the equation

$$\sum_{i=1}^{\ell} C_i \alpha_i^n \beta_i^m = 0 \tag{1}$$

has finitely many solutions in integers n, m. Moreover there is an explicit upper bound on the number of solutions in terms of ℓ and the degree of the α_i, β_i, C_i.

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If $u_n = 0$ and n has decomposition $n = qP + a$ then q, a solve a "companion equation".

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Proposition

Let $\langle u_n \rangle$ be a non-degenerate simple LRS. Then there is a computable upper bound on the set $\{n \in \mathcal{U} : u_n = 0\}$.

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Theorem

The set $\mathcal{U}$ has positive lower density.

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Theorem (Sieve)

Let $a_1, a_2, b_1, b_2 \in \mathbb{Z}$ be such that $|a_1 a_2(a_1 b_2 - a_2 b_1)| \neq 0$. Then

$$|\{t \leq X : a_1 t + b_1, a_2 t + b_2 \text{ both prime}\}| \ll \frac{X}{(\log X)^2}.$$

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"The exact formulation of Cramér's conjecture has been called into question. It is still probably true that for all $c > 2$,

$$\max_{p_n \leq x}(p_{n+1} - p_n) = O(\log^c x)\text{ "}$$

Adelman and McCurley

Heuristics for the Skolem Problem

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Theorem

There is a set $\mathcal{U}$ of positive lower density for which there is an algorithm that takes as input a non-degenerate LRS and outputs its set of zeros in $\mathcal{U}$.

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- Cramér heuristic suggests that $\mathcal{U}$ has density one.