The Stochastic Arrival Problem

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Reachability Problems, October 2022

The ARRIVAL Problem

ARRIVAL ([DGKMW,17])

We are given the following input:

- A directed graph (V, E),
- A source vertex $s \in V$,
- A target vertex $t \in V$,
- For each $v \in V$ an *ordered list* $Ord(v) := (w_0, \ldots, w_{k_v-1})$ *where each* $(v, w_i) \in E$ *is a directed edge out of* v*.*

- Decide whether or not we reach t starting from s using a **"switching walk"**.

Switching Walk

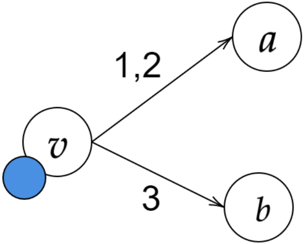
A switching walk maintains a counter q_v for every vertex $v \in V$, initialized to 0, which counts how many times that vertex has been visited thusfar during the walk.

If the switching walk starting at s is currently at vertex v, then the next edge used by the switching walk is to w_j, where $j \equiv q_v \pmod{k_v}$.

We can view the vector of counter values $\langle q_v \pmod{k_v} \mid v \in V \rangle$ to be our current "**switching state**" and call the (exponentially large) set of possible switching states Q.

ARRIVAL instances can be viewed as a succinct presentation of exponentially larger reachability problem on a directed graph without Switching, with state space $V \times Q$.

A Quick Example



$$Ord(v) = (a, a, b)$$

$$q_v = 0$$

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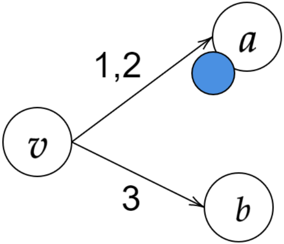

$$Ord(v) = (a, a, b)$$

$$q_v = 1$$

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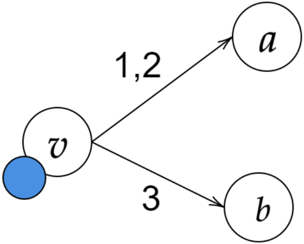

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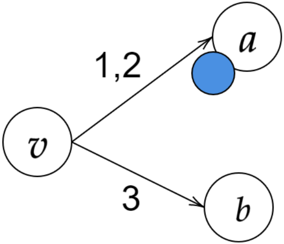

$$Ord(v) = (a, a, b)$$

$$q_v = 2$$

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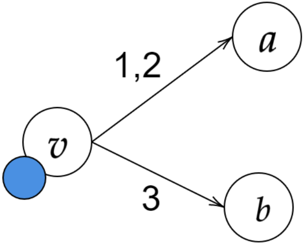

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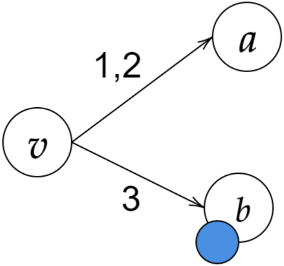

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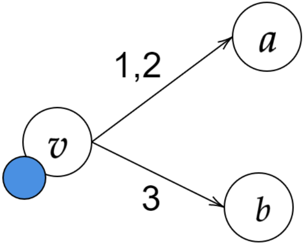

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Prior Work on ARRIVAL and its extension to Switching Games

Known complexity results for ARRIVAL

- ARRIVAL is in NP $\cap$ coNP, but not known to be in P ([DGKMW'17])
- It is in PLS, CLS and UniqueEOPL ([K'17],[GHHKMS'18],[FGMS'19])
- There is a sub-exponential algorithm for ARRIVAL. ([GHH'21])

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Reachability Switching Games

Fearnley, Gairing, Mnich & Savani (2021) considered 1 and 2 player extensions, where nodes are either controlled by one of the players, or are **switching nodes** following a given switching order on outgoing edges.

Our New Work

Randomisation & Switching

- We look at extensions to ARRIVAL where there are also **random nodes** where the walk advances randomly or are **switching nodes** following the given switching order.
- We also consider extensions with players in addition to random and switching nodes.
- These correspond to succinct representations of exponentially-large Markov Chains, Markov Decision Processes and Simple Stochastic Games.

Combining randomisation nodes with Switching and player nodes.

Definition (B-Arrival problem instance)

We partition the nodes V into four different types:

- V_R : Walk makes a uniformly random choice over outgoing edges
- V_S : Switching behaviour determined by Ord
- V_1 : Controlled by player 1, who is trying to reach t
- V_2 : Controlled by player 2, who is trying to avoid t

For some set $B \subseteq \{R, S, 1, 2\}$ we use B-Arrival to describe instances only containing node types in B, e.g., $\{R, S, 1\}$-Arrival denotes instances which may contain nodes of type R, S and 1, but none of type 2.

Stochastic Switching Game Problems

Definition (value)

We define the **value of the instance** as follows. Let *Reach* be the event of eventually hitting t starting at s, and let σ_1 and τ_2 range over strategies for player 1 and 2, respectively. Define the **value** of the stochastic switching game as:

$$val(G) := \max_{\sigma_1} \min_{\tau_2} \mathbb{P}_{\sigma_1,\tau_2}(Reach)$$

Problems

Let $p \in (0,1)$ be a given (rational) probability input. We define three variants of B-Arrival decision problems:

- B-`Arrival-Quant`: Decide whether $val(G) > p$.
- B-`Arrival-Qual-0`: Decide whether $val(G) > 0$.
- B-`Arrival-Qual-1`: Decide whether $val(G) = 1$.

Prior Results on Reachability Switching Games

Theorem ([FGMS'21])

$\{S, 1\}$-Arrival is NP-complete.

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Theorem ([FGMS'21])

$\{S, 1, 2\}$-Arrival-Quant is PSPACE-hard.

Theorem ([FGMS'21])

There is an EXPTIME algorithm to decide $\{S, 1, 2\}$-Arrival.

Our Main Results

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$\{R, S\}$-Arrival-Quant is PP-hard.

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$\{R, S, 1\}$-Arrival-Quant is PSPACE-hard.

$\{R, S, 1\}$-Arrival-Quant

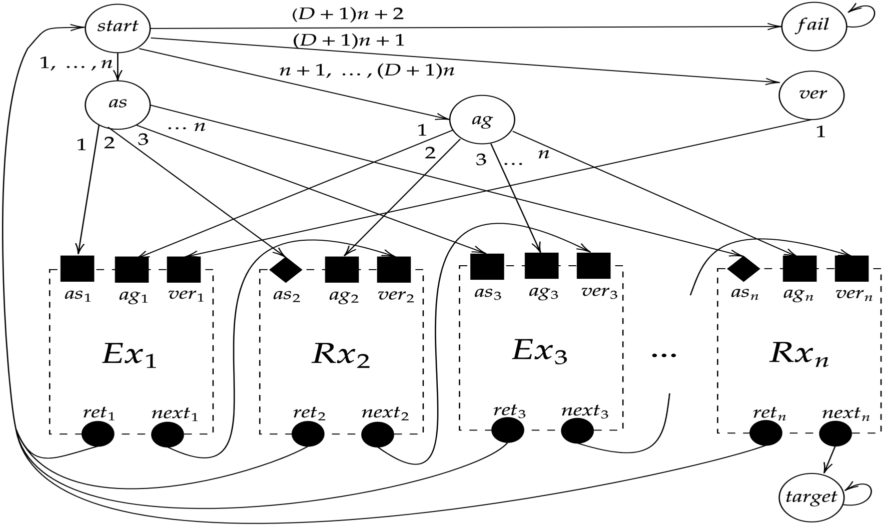
We reduce from a PSPACE-hard problem defined by Papadimitriou (85):

Definition (SSAT)

- Given a 3CNF formula φ
- Reading Я as "for uniformly random"
- Decide if:

$$\exists x_1 Я x_2 \exists x_3 \dots Я x_n [\mathbb{P}(\varphi(x_1, \dots, x_n) = \top) > \frac{1}{2}] \tag{1}$$

Reduction from SSAT



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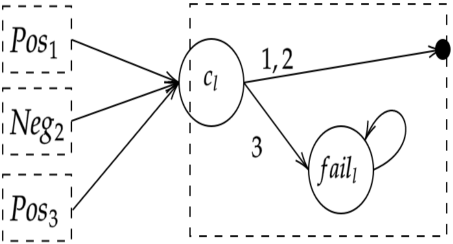
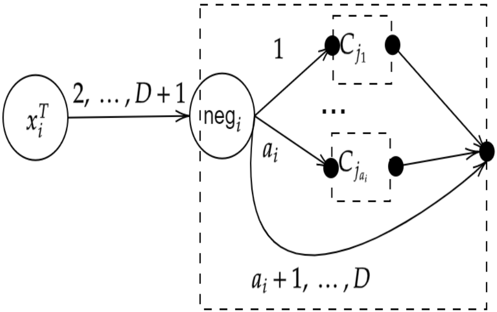


$C_l : x_1 \vee \neg x_2 \vee x_3$

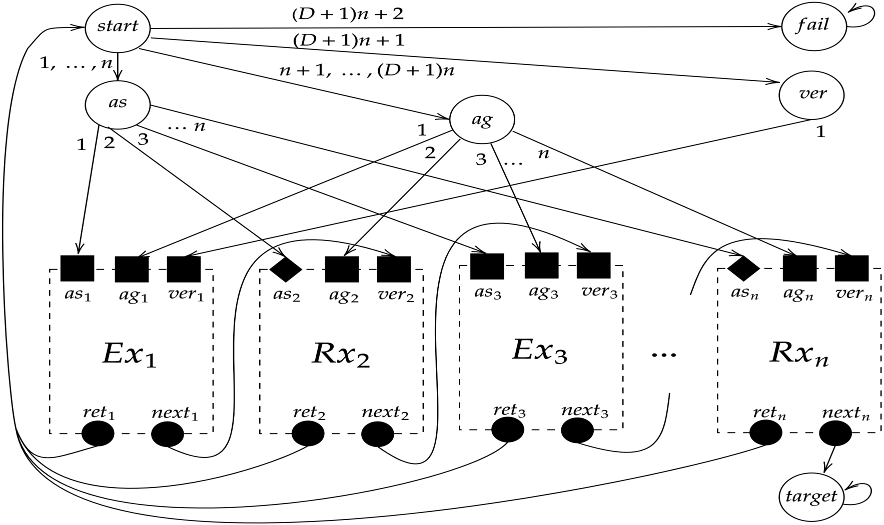
Reduction from SSAT



$C_{j_1}, \ldots, C_{j_{a_i}}$ are clauses containing the literal $\neg x_i$

D suff. large constant

Reduction from SSAT



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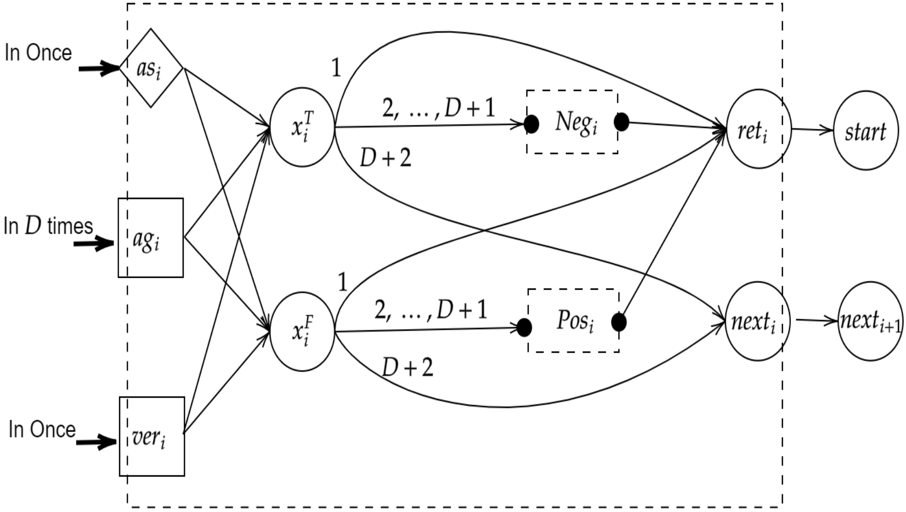

A PSPACE Algorithm for $\{R, S\}$-Arrival-Quant

- We can use $\{R, S\}$-`Arrival-Qual-0` as a subroutine within PSPACE to eliminate states $(v, q) \in V \times Q$ that have no paths to the target.
- We can check reachability in the exponentially sized Markov Chain to eliminate states $(v, q) \in V \times Q$ that can't be reached from the start.
- We can then compute in PSPACE elements of a modified transition matrix P for the exponential Markov Chain, with some branches pruned to make the chain irreducible.
- We can compute a given bit of an entry of P^{-1}, thus finding the leading bit of the hitting probability.
- Showing the $val(G) \geq \frac{1}{2}$ and $val(G) > \frac{1}{2}$ are equivalent.

New Results

A full list of results shown in the paper:

Problem Name	Complexity
$\{S, 2\}$-`Arrival`	NP-complete
$\{R, S\}$-`Arrival-Qual-0`	NP-complete
$\{R, S\}$-`Arrival-Qual-1`	coNP-complete
$\{R, S\}$-`Arrival-Quant`	PP-hard, in PSPACE
$\{R, S, 1\}$-`Arrival-Qual-0`	NP-complete
$\{R, S, 1\}$-`Arrival-Qual-1`	coNP-hard, in EXPTIME
$\{R, S, 1\}$-`Arrival-Quant`	PSPACE-hard, in EXPTIME
$\{R, S, 2\}$-`Arrival-Qual-0`	equiv $\{S, 1, 2\}$-`Arrival`
$\{R, S, 2\}$-`Arrival-Qual-1`	in EXPTIME
$\{R, S, 2\}$-`Arrival-Quant`	PSPACE-hard, in EXPTIME
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$\{R, S, 1, 2\}$-`Arrival-Qual-1`	in NEXPTIME $\cap$ coNEXPTIME
$\{R, S, 1, 2\}$-`Arrival-Quant`	PSPACE-hard, in NEXPTIME $\cap$ coNEXPTIME